

A local limit theorem for triple connections in subcritical Bernoulli percolation

M. Campanino*, M. Gianfelice*
 Dipartimento di Matematica
 Università degli Studi di Bologna
 P.zza di Porta San Donato, 5 I-40127

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Abstract

We prove a local limit theorem for the probability of a site to be connected by disjoint paths to three points in subcritical Bernoulli percolation on \mathbb{Z}^d , $d \geq 2$ in the limit where their distances tend to infinity.

1 Introduction and results

The asymptotic behaviour of the connection function for Bernoulli sub-critical percolation on d -dimensional lattices and of two points correlation functions of finite range Ising models above critical temperature has been recently completely proved to agree with that predicted by Ornstein and Zernike ([CI], [CIV], see also [AL], and [CCC] for some previous results and [BF] for some results for extreme values of parameters). The arguments of [CI] and [CIV] follow a general scheme that is exposed in [CIV1]. A natural question that arises is how higher order percolation or correlation functions behave for these models. It is natural to

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start by addressing this problem in the simplest case, i.e. for triple connection functions in Bernoulli subcritical percolation on d -dimensional lattices. This analysis is carried out in this work. It turns out that the techniques developed in [CI] and [CIV], plus some extra ideas, allow to obtain the asymptotic behaviour. Interestingly and luckily enough, some techniques introduced in [CIV] for the Ising models result useful for our work, though for somewhat different reasons. Besides the asymptotic behaviour of the probability of triple connections we obtain a local limit theorem for the positions of points from which three disjoint paths start and give rise to triple connections. It is worth to observe that these positions are not decomposed in a natural way as sums of random variables, as it is most common in local limit theorems.

We consider a Bernoulli bond percolation process on \mathbb{Z}^d , $d \geq 2$, in the subcritical regime ($p < p_c(d)$). A basic result, established by Menshikov [M] and by Aizenman and Barsky [AB] with different methods, states that, for $p < p_c(d)$, connection functions decay exponentially in every direction. Using the FKG inequality it can be shown that, given a point n of the lattice, the probability $\mathbb{P}_p\{0 \leftrightarrow n\}$ that n is connected to the origin 0 (i.e. that there exists a chain of open bonds leading from the origin to n) is bounded from above by $e^{-\xi_p(n)}$, where

$$\xi_p(x) := - \lim_{N \uparrow \infty} \frac{1}{N} \log \mathbb{P}_p\{0 \leftrightarrow [xN]\}. \quad (1)$$

ξ_p , is always defined and is a finite, convex, homogeneous-of-order-one function on \mathbb{R}^d , invariant under permutation and reflection across coordinate hyperplanes. For $\|x\| = 1$, ξ_p goes by the name of *inverse connection length* in the direction x .

Let us denote with (\cdot, \cdot) the scalar product in \mathbb{R}^d , and with $\|\cdot\| := \sqrt{(\cdot, \cdot)}$ the associated Euclidean norm. It has been proved by Hammersley ([G] Theorem 5.1) that if $p < p_c(d)$ there exists a strictly positive function $c_-(p)$ such that

$$\xi_p(x) \geq c_-(p) \|x\| \quad x \in \mathbb{R}^d \setminus \{0\}, \quad (2)$$

while from Harris inequality it follows that

$$\xi_p(x) \leq c_+(p) \|x\| \quad x \in \mathbb{R}^d \setminus \{0\}, \quad (3)$$

which implies that the inverse correlation length is an equivalent norm on \mathbb{R}^d .

Following a previous work by [CCC], where only the axes directions were considered, recently Campanino and Ioffe showed ([CI] Theorem A) that if the lattice dimension d is larger than or equal to 2, uniformly in $x \in \mathbb{S}^{d-1}$, the correct asymptotics for the *connectivity function* $\mathbb{P}_p\{0 \leftrightarrow [Nx]\}$ for $p < p_c(d)$ is given by

$$\mathbb{P}_p\{0 \leftrightarrow [Nx]\} = \frac{\Psi_p(x)}{\sqrt{(2\pi N)^{d-1}}} e^{-\xi_p([Nx])} (1 + o(1)), \quad (4)$$

where Ψ_p is a positive real analytic function on \mathbb{S}^{d-1} .

Let

$$\mathbf{U}^p := \{x \in \mathbb{R}^d : \xi_p(x) \leq 1\} \quad (5)$$

be the unit ball in the ξ_p -norm (ξ_p -ball), then any ξ_p -ball will be denoted by

$$a\mathbf{U}^p := \{x \in \mathbb{R}^d : \xi_p(x) \leq a\} \quad a \in \mathbb{R}^+. \quad (6)$$

We also introduce the polar body of \mathbf{U}^p

$$\mathbf{K}^p := \bigcap_{x \in \mathbb{S}^{d-1}} \{t \in \mathbb{R}^d : (t, x) \leq \xi_p(x)\} \quad (7)$$

Then, given any $x \in \mathbb{R}^d$, the set of vectors $t \in \partial\mathbf{K}^p$ meeting the equality

$$(t, x) = \xi_p(x) \quad (8)$$

are said to be polar to x . It has been shown ([CI] Lemma 4.3) that both $\partial\mathbf{U}^p$ and $\partial\mathbf{K}^p$ are strictly convex analytic surfaces with gaussian curvature bounded away from zero, so there exists only one point $t_x \in \partial\mathbf{K}^p$ satisfying the equality (8).

In this paper, using the tools introduced in [CI], we will analyse the probability that three distinct points of the lattice are connected through disjoint open paths, in the limit as their mutual distance tends to infinity. To this aim we need to introduce some additional notation.

For $x \in \mathbb{R}^d$, let us denote by $[x]$ the vector $([x_1], \dots, [x_d])$ and define

$$X_3 := \{(x_1, x_2, x_3) \in \mathbb{R}^{3d} : x_i \neq x_j \text{ if } i \neq j; i, j = 1, 2, 3\}. \quad (9)$$

Hence, for $\mathbf{x} \in X_3$, we define

$$\varphi_{p,\mathbf{x}}(x) := \sum_{i=1}^3 \xi_p(x - x_i). \quad (10)$$

$\varphi_{p,\mathbf{x}}(x)$ is easily seen to be a convex function whose unique minimum, which is a function of \mathbf{x} , we will denote by $x_0(\mathbf{x})$. In the following, we will consider only those elements of X_3 satisfying the further condition:

$$u_i := \sum_{j \neq i} \nabla \xi_p(x_j - x_i) \notin \mathbf{K}^p \quad \forall i = 1, 2, 3. \quad (11)$$

The geometrical meaning of (11) relies on the fact that, given $\mathbf{x} \in X_3$, this condition prevents $x_0(\mathbf{x})$ to coincide with one of the entries of \mathbf{x} . Let then X'_3 be the subset of X_3 whose elements satisfy (11) and, given three distinct vertices n_1, n_2, n_3 of the lattice, let:

•

$$E(n_1, n_2, n_3) = E(\mathbf{n}) \quad (12)$$

be the event that n_1, n_2, n_3 are connected by an open cluster;

•

$$F(k; n_1, n_2, n_3) = F(k; \mathbf{n}) \quad k \in \mathbb{Z}^d \quad (13)$$

be the event that k is connected by three disjoint self-avoiding open paths $\gamma_1, \gamma_2, \gamma_3$ to n_1, n_2, n_3 respectively.

Then we have

Theorem 1 *Let $\mathbf{x} \in X'_3$, $y \in \mathbb{R}^d$ and let N vary over the integers. If we denote by $x_0(\mathbf{x})$ the minimizing point of the function $\varphi_{p,\mathbf{x}}$, then, for $d \geq 2$ and $p < p_c(d)$,*

$$\begin{aligned} & \mathbb{P}_p \left[F \left(\left[x_0(\mathbf{x}) N + y \sqrt{N} \right]; [N\mathbf{x}] \right) \mid E([N\mathbf{x}]) \right] = \\ & \Phi_p(\mathbf{x}) \frac{\sqrt{\det H_\varphi(x_0(\mathbf{x}), \mathbf{x}; p)}}{(2\pi N)^{\frac{d}{2}}} \exp \left[-\frac{(y, H_\varphi(x_0(\mathbf{x}), \mathbf{x}; p) y)}{2} \right] (1 + o(1)), \end{aligned} \quad (14)$$

where $H_\varphi(x_0(\mathbf{x}), \mathbf{x}; p)$ is the Hessian matrix of the function $\varphi_{p,\mathbf{x}}$ evaluated at $x_0(\mathbf{x})$, and $\Phi_p(\mathbf{x})$ is an analytic function on X'_3 .

Remark 2 *For any $\varepsilon \in (0, \frac{1}{2})$ and any $\beta \in (0, \frac{1}{2})$, let*

$$F_{\varepsilon,\beta}(k_1, k_2; [N\mathbf{x}]) := F(k_1; [Nx]) \cap F(k_2; [Nx]) \cap \left\{ k_1, k_2 \in N^{\frac{1}{2}+\varepsilon} \mathbf{U}^p(Nx_0(\mathbf{x})) \cap \mathbb{Z}^d : \right. \\ \left. ||k_1 - k_2|| > N^\beta \right\}$$

be the event that two lattice's points k_1 and k_2 , belonging to a ξ_p -neighborhood of $[Nx_0(\mathbf{x})]$ of radius $N^{\frac{1}{2}+\varepsilon}$ and whose mutual distance is larger than N^β , are connected to $[Nx_1], [Nx_2], [Nx_3]$ by three disjoint self-avoiding open paths. As a byproduct, in the proof of Theorem 1 we also get that there exists a positive constant c'' such that,

$$\mathbb{P}_p[F_{\varepsilon,\beta}(k_1, k_2; [N\mathbf{x}])] \leq e^{-N\varphi_{p,\mathbf{x}}(x_0(\mathbf{x})) - c''N^{\beta \wedge 2\varepsilon}}. \quad (15)$$

2 Local limit theorem

2.1 Preliminary results

Let us define

$$\mathcal{H}_y^t := \{x \in \mathbb{R}^d : (t, x) = (t, y)\} \quad y \in \mathbb{R}^d \quad (16)$$

to be the $(d-1)$ -dimensional hyperplane in \mathbb{R}^d orthogonal to the vector t passing through a point $y \in \mathbb{R}^d$ and the corresponding halfspaces

$$\mathcal{H}_y^{t,-} := \{x \in \mathbb{R}^d : (t, x) \leq (t, y)\}, \quad (17)$$

$$\mathcal{H}_y^{t,+} := \{x \in \mathbb{R}^d : (t, x) \geq (t, y)\}. \quad (18)$$

Then we have

Lemma 3 *For any $\mathbf{x} \in X'_3$, $\varphi_{p,\mathbf{x}}$ is a strictly convex function on a neighborhood of $x_0(\mathbf{x})$, where it is lower bounded by a strictly positive quadratic form of $x - x_0(\mathbf{x})$.*

Proof. Setting $y = x - x_0(\mathbf{x})$ we consider $\varphi_{p,\mathbf{y}}(y) = \sum_{i=1}^3 \xi_p(y - y_i)$, where $y_i = x_i - x_0(\mathbf{x})$. Let $t_i \in \partial \mathbf{K}^p$ be the polar point to y_i . By the convexity of ξ_p and [CI] Lemma 4.4, there exist positive constants c', c such that, for any $z \in \mathbb{R}^d$ satisfying, $(z, t_i) = (y_i, t_i) = \xi_p(y_i)$ and $\|z - y_i\| \leq c'$,

$$\xi_p(z) \geq (t_i, z) + c\|z - y_i\|^2. \quad (19)$$

Hence, for any $y \in \mathbb{R}^d$ such that $\|y\| \leq c'$, setting $z = y_i - y$, we get

$$\xi_p(y - y_i) - \xi_p(y_i) \geq -(\nabla \xi_p(y_i), y) + c\|P_i^\perp y\|^2 \quad i = 1, 2, 3, \quad (20)$$

where, $\forall i = 1, 2, 3$, P_i^\perp is the orthogonal projector on $\mathcal{H}_0^{t_i}$. Summing up, since by the definition of $x_0(\mathbf{x})$,

$$\sum_{i=1}^3 \nabla \xi_p(y_i) = \sum_{i=1}^3 \nabla \xi_p(x_0(\mathbf{x}) - x_i) = 0, \quad (21)$$

we get

$$\varphi_{p,\mathbf{y}}(y) - \varphi_{p,\mathbf{y}}(0) \geq c \sum_{i=1}^3 \|P_i^\perp y\|^2. \quad (22)$$

The right hand side of the last expression can never be zero for $y \neq 0$ because, $\forall i = 1, 2, 3$, the hyperplanes $\mathcal{H}_0^{t_i}$ have codimension one and conditions (11) and (21) prevent the vectors $\nabla \xi_p(x_0(\mathbf{x}) - x_i)$, $i = 1, 2, 3$, from being parallel. ■

For $l \geq 1$, let $\mathbf{C}_{\{k_1, \dots, k_l\}}$ denote the common open cluster of the points $k_1, \dots, k_l \in \mathbb{Z}^d$, provided it exists, and let $t \in \partial \mathbf{K}^p$. Given two points k_i, k_j such that $(k_i, t) \leq (k_j, t)$, we denote by $\mathbf{C}_{\{k_i, k_j\}}^t$ the cluster of k_i and k_j inside the strip $\mathcal{S}_{\{k_i, k_j\}}^t := \mathcal{H}_{k_i}^{t,+} \cap \mathcal{H}_{k_j}^{t,-}$.

First we estimate the probability that, $\forall i = 1, 2, 3$, the points $[Nx_i]$ are connected through three disjoint open paths to a point whose distance from $x_0([N\mathbf{x}])$ is larger than N^α with $\alpha \in (\frac{1}{2}, 1)$.

For any $\mathbf{x} \in X'_3$, let $\mathbf{C}_{[N\mathbf{x}]} = \mathbf{C}_{\{[Nx_1], [Nx_2], [Nx_3]\}}$ and

$$A_{\alpha, N}(\mathbf{x}) := \left\{ \exists n \in \mathbf{C}_{[N\mathbf{x}]} : n \xleftrightarrow{\gamma_i} [Nx_i], \gamma_i \cap \gamma_j = n, i, j = 1, 2, 3, i \neq j; \|n - x_0([N\mathbf{x}])\| \geq N^\alpha \right\} \quad (23)$$

be the event that the lattice points $[Nx_i]$ are connected through three disjoint open paths to a point n whose distance from $x_0([N\mathbf{x}])$ is larger than or equal to N^α . We have

Proposition 4 *For any $\mathbf{x} \in X'_3$ and $\alpha > \frac{1}{2}$,*

$$\mathbb{P}_p[A_{\alpha, N}(\mathbf{x})] \leq e^{-\varphi_{p, [N\mathbf{x}]}(x_0([N\mathbf{x}]))} e^{-c_1 N^{2\alpha-1}}, \quad (24)$$

with c_1 a positive constant.

Proof. By the BK inequality (see e.g. [G])

$$\mathbb{P}_p[A_{\alpha, N}(\mathbf{x})] \leq e^{-\varphi_{p, [N\mathbf{x}]}(x_0([N\mathbf{x}]))} \sum_{n \in \mathbb{Z}^d : \|n - x_0([N\mathbf{x}])\| \geq N^\alpha} e^{-[\varphi_{p, [N\mathbf{x}]}(n) - \varphi_{p, [N\mathbf{x}]}(x_0([N\mathbf{x}]))]}. \quad (25)$$

The convexity of $\varphi_{p, [N\mathbf{x}]}$ implies that given $z \in \mathbb{R}^d$, for any point y lying on the segment between z and $x_0([N\mathbf{x}])$, we have

$$\varphi_{p, [N\mathbf{x}]}(z) - \varphi_{p, [N\mathbf{x}]}(x_0([N\mathbf{x}])) \geq \frac{\|z - x_0([N\mathbf{x}])\|}{\|y - x_0([N\mathbf{x}])\|} (\varphi_{p, [N\mathbf{x}]}(y) - \varphi_{p, [N\mathbf{x}]}(x_0([N\mathbf{x}]))). \quad (26)$$

Since ξ_p is a homogeneous function of order one, its Hessian matrix $H_\xi(\cdot; p)$ is a homogeneous function of order -1 . Hence, choosing y such that $\|y - x_0([N\mathbf{x}])\| = N^\alpha$, $\forall i = 1, 2, 3$, $\|y - [Nx_i]\| \geq N^\alpha$ and by (22) there exists a positive constant c_2 such that

$$\varphi_{p, [N\mathbf{x}]}(y) - \varphi_{p, [N\mathbf{x}]}(x_0([N\mathbf{x}])) \geq c_2 N^{2\alpha-1}. \quad (27)$$

Furthermore, for any z outside of a neighbourhood of $[Nx_i]$, $i = 1, 2, 3$,

$$\left(H_\varphi \left(z, \frac{[N\mathbf{x}]}{N}; p \right) \right)_{i,j} = (H_\varphi(z, \mathbf{x}; p))_{i,j} + O\left(\frac{1}{N}\right), \quad i, j = 1, \dots, d. \quad (28)$$

Substituting (26) and (27) into (25), for values of N large enough, we can bound the r.h.s. of (25) by

$$\begin{aligned} \sum_{n \in \mathbb{Z}^d : \|n - x_0([N\mathbf{x}])\| \geq N^\alpha} e^{-c_2 N^{\alpha-1} \|n - x_0([N\mathbf{x}])\|} &\leq c_3 \int_{\{x \in \mathbb{R}^d : \|x\| \geq N^\alpha\}} dx e^{-c_2 N^{\alpha-1} \|x\|} \\ &\leq \frac{C_4}{N^{(\alpha-1)d}} \int_{c_2 N^{2\alpha-1}}^{\infty} r^{(d-1)} e^{-r} dr \leq c_4 N^{d(1-\alpha)} e^{-\frac{c_2}{2} N^{2\alpha-1}}. \end{aligned} \quad (29)$$

■

2.2 Renewal structure of connectivities

Given $t \in \partial \mathbf{K}^p$ and a positive number $\eta < 1$, we define the set (*surcharge cone*)

$$\mathcal{C}_\eta(t) := \{x \in \mathbb{R}^d : (t, x) \geq (1 - \eta) \xi_p(x)\}. \quad (30)$$

We now follow [CI] and [CIV]. Let e be the first of the unit vectors e_1, \dots, e_d in the direction of the coordinate axis such that (t, e) is maximal and let x_t denote the element of $\partial \mathbf{U}^p$ polar to t .

Definition 5 $k, n \in \mathbb{Z}^d$ are called h_t -connected if

1 - n and k are connected in $\mathcal{S}_{\{k, n\}}^t$;

2 -

$$\mathbf{C}_{\{k, n\}}^t \cap \mathcal{S}_{\{k, k+e\}}^t = \{k, k+e\}, \quad \mathbf{C}_{\{k, n\}}^t \cap \mathcal{S}_{\{n-e, n\}}^t = \{n-e, n\}. \quad (31)$$

Moreover, denoting by $\left\{k \overset{h_t}{\longleftrightarrow} n\right\}$ the event that n and k are h_t -connected, we set

$$h_t(k, n) := \mathbb{P}_p \left\{k \overset{h_t}{\longleftrightarrow} n\right\}. \quad (32)$$

Notice that, by translation invariance, $h_t(k, n) = h_t(n - k, 0)$ so in the sequel we will denote it simply by $h_t(n - k)$. We also define by convention $h_t(0) = 1$.

Definition 6 Let $k, n \in \mathbb{Z}^d$ be connected. The points $b \in \mathbf{C}_{\{k, n\}}^t$ such that:

1 - $(t, k+e) \leq (t, b) \leq (t, n-e)$;

2 - $\mathbf{C}_{\{k, n\}}^t \cap \mathcal{S}_{\{b-e, b+e\}}^t = \{b-e, b+e\}$;

are said to be t -break points of $\mathbf{C}_{\{k,n\}}$. The collection of such points, which we remark is a totally ordered set with respect to the scalar product with t , will be denoted by $\mathbf{B}^t(k, n)$.

Definition 7 Let $k, n \in \mathbb{Z}^d$ be h_t -connected. If $\mathbf{B}^t(k, n)$ is empty, then n and k are said to be f_t -connected and the corresponding event is denoted by $\left\{k \xleftrightarrow{f_t} n\right\}$. We then set $f_t(n - k) := \mathbb{P}_p \left\{k \xleftrightarrow{f_t} n\right\}$.

We define by convention $f_t(0) = 0$. We now introduce a particular subset of t -break points (inspired by section 2.6 of [CIV]) which will provide a decomposition of the event $E([N\mathbf{x}])$ into suitable disjoint events.

Let K be a positive constant that will be chosen sufficiently large.

Definition 8 $k, n \in \mathbb{Z}^d$ are said to be $h_t^{\eta, K}$ -connected and the corresponding event is denoted by $\left\{k \xleftrightarrow{h_t^{\eta, K}} n\right\}$, if n and k are h_t -connected and satisfy the following conditions:

$$1 - n \in k + \mathcal{C}_\eta(t);$$

$$2 - \mathbf{C}_{\{k,n\}}^t \subseteq (k - Kx_t) + \mathcal{C}_\eta(t).$$

As for the other connection functions we put $h_t^{\eta, K}(0) = h_t(0) = 1$.

Definition 9 Let $k, n \in \mathbb{Z}^d$ be connected. We define b_1 to be the element of the set

$$\left\{l \in \mathbf{C}_{\{k,n\}} \cap \mathcal{H}_k^{t,+} : \mathbf{C}_{\{l,n\}} \cap \mathcal{H}_l^{t,+} \subseteq (l - Kx_t) + \mathcal{C}_\eta(t)\right\} \quad (33)$$

satisfying:

$$a - \mathbf{C}_{\{k,n\}} \cap \mathcal{S}_{\{b_1 - e, b_1\}}^t = \{b_1 - e, b_1\};$$

$$b - (b_1 - k, t) \text{ is maximal.}$$

Given b_j ($j \geq 1$), we denote by b_{j+1} the first t -break point of $\mathbf{C}_{\{k,n\}}^t$ following b_j satisfying the following conditions:

$$1 - b_j \in b_{j+1} + \mathcal{C}_\eta(t);$$

$$2 - \xi_p(b_j - b_{j+1}) \geq \frac{2K}{\eta};$$

$$\mathcal{B} - \mathbf{C}_{\{b_{j+1}, b_j\}}^t \subseteq (b_{j+1} - Kx_t) + \mathcal{C}_\eta(t);$$

provided it exists. We will call these points (η, K, t) -break points and denote their collection by $\mathbf{B}^t(k, n; \eta, K)$.

Definition 10 Any two distinct points $k, n \in \mathbb{Z}^d$ are said to be $f_t^{\eta, K}$ -connected if they are $h_t^{\eta, K}$ -connected and $\mathbf{B}^t(k, n; \eta, K) = \emptyset$.

Clearly, $f_t^{\eta, K}(0) = f_t(0) = 0$.

Lemma 11 Let $t \in \partial \mathbf{K}^p$ and let $k, n \in \mathbb{Z}^d$, with $(t, n - k) > 0$, be connected. It is possible to choose $\eta \in (0, 1)$ small enough and K sufficiently large such that, if

$$\mu := \max\{j \geq 2 : b_j \in \mathbf{B}^t(k, n; \eta, K)\}, \quad (34)$$

then $\mathbf{C}_{\{b_{\mu-1}, n\}} \cap \mathcal{H}_{b_{\mu-1}}^{t,+} \subset b_\mu + \mathcal{C}_{2\eta}(t)$.

Proof. $\forall m \in \mathbf{C}_{\{b_{\mu-1}, n\}} \cap \mathcal{H}_{b_{\mu-1}}^{t,+}$ we set

$$l = l(m) := \min\{1 \leq j \leq \mu - 1 : (b_j - k, t) \leq (m - k, t)\} \quad (35)$$

and, since $1 \leq l \leq \mu - 1$, we consider the following cases:

1. If $l = \mu - 1$, by Definition 9, $m \in b_{\mu-1} - Kx_t + \mathcal{C}_\eta(t)$ and $b_{\mu-1} \in b_\mu + \mathcal{C}_\eta(t)$. Hence,

$$\begin{aligned} (m - b_\mu + Kx_t, t) &= (m - b_{\mu-1} + Kx_t + b_{\mu-1} - b_\mu, t) = \\ &= (m - b_{\mu-1} + Kx_t, t) + (b_{\mu-1} - b_\mu, t) \geq \\ &= (1 - \eta) \xi_p(m - b_{\mu-1} + Kx_t, t) + (1 - \eta) \xi_p(b_{\mu-1} - b_\mu) \geq \\ &= (1 - \eta) \xi_p(m - b_\mu + Kx_t, t) \end{aligned} \quad (36)$$

that is $m \in b_\mu - Kx_t + \mathcal{C}_\eta(t)$. If $m \in b_\mu + \mathcal{C}_\eta(t)$, then the thesis is verified. Otherwise, $(m - b_\mu, t) \leq (1 - \eta) \xi_p(m - b_\mu)$. Therefore,

$$\begin{aligned} \xi_p(m - b_\mu) &\geq \frac{(m - b_\mu, t)}{1 - \eta} \geq \frac{(m - b_{\mu-1}, t)}{1 - \eta} + \xi_p(b_{\mu-1} - b_\mu) \\ &\geq \xi_p(b_{\mu-1} - b_\mu). \end{aligned} \quad (37)$$

Moreover,

$$\begin{aligned} (m - b_\mu, t) &= (m - b_\mu + x_t K - x_t K, t) = (m - b_\mu + x_t K, t) - K \geq \\ &= (1 - \eta) \xi_p(m - b_\mu + x_t K) - K \geq \end{aligned} \quad (38)$$

$$\begin{aligned} &= (1 - \eta) \xi_p(m - b_\mu) - (1 - \eta) K - K = \\ &= (1 - 2\eta) \xi_p(m - b_\mu) + \eta \xi_p(m - b_\mu) - 2K + \eta K \geq \\ &= (1 - 2\eta) \xi_p(m - b_\mu) + \eta \xi_p(b_\mu - b_{\mu-1}) - 2K, \end{aligned} \quad (39)$$

but, by condition 2 of Definition 9, $\xi_p(b_{\mu-1} - b_\mu) \geq \frac{2K}{\eta}$. Thus $(m - b_\mu, t) \geq (1 - 2\eta) \xi_p(m - b_\mu)$.

2. If $1 \leq l \leq \mu - 2$,

$$m - b_\mu = b_{l+1} - b_\mu + m - b_{l+1} = m - b_{l+1} + \sum_{j=1}^{\mu-l-1} (b_{l+j} - b_{l+j+1}), \quad (40)$$

then, since by the previous case $m \in b_{l+1} + \mathcal{C}_{2\eta}(t)$ and by condition 1 of Definition 9 $b_{l+j} - b_{l+j+1} \in \mathcal{C}_\eta(t) \subset \mathcal{C}_{2\eta}(t)$, the r.h.s. of (40), as a sum of elements of $\mathcal{C}_{2\eta}(t)$, also belongs to $\mathcal{C}_{2\eta}(t)$.

■

For any $b \in \mathbb{Z}^d$, it is easy to see that

$$f_t^{\eta,K}(b) \leq h_t^{\eta,K}(b) \leq h_t(b) \leq e^{-\xi_p(b)}. \quad (41)$$

From the previous definitions it follows that $h_t^{\eta,K}(n)$ satisfies a renewal equation analogous to the one satisfied by h_t -connected points given in [CI] (4.3), i.e.

$$h_t^{\eta,K}(n) = \sum_{b \in \mathbb{Z}^d} h_t^{\eta,K}(b) f_t^{\eta,K}(n - b). \quad (42)$$

Furthermore, it can be shown that $h_t^{\eta,K}([Nx])$, for x in a neighbourhood of the dual point of t , satisfies the same asymptotic behaviour of $h_t([Nx])$ (see [CI] Lemma 4.5). That is, for any $\eta \in (0, 1)$ and K large enough,

$$h_t^{\eta,K}([Nx]) = \frac{\Lambda_p\left(\frac{x}{\|x\|}, t\right)}{\sqrt{2\pi N^{d-1} \|x\|^{d-1}}} e^{-\xi_p([Nx])} (1 + o(1)), \quad (43)$$

where $\Lambda_p(\cdot, t)$ is an analytic function on \mathbb{S}^{d-1} (different from that relative to h_t appearing in [CI] (4.18)). The proof of this assertion relies on arguments similar to the ones used in [CI] to prove the Ornstein-Zernike theory for the connectivity function and so it will be omitted.

Definition 12 *Let $k, n \in \mathbb{Z}^d$ be connected. Then:*

1 - k, n are called \bar{h}_t -connected and the corresponding event is denoted by $\left\{k \xleftrightarrow{\bar{h}_t} n\right\}$, if

$$\mathbf{C}_{\{k,n\}} \cap \mathcal{S}_{\{n-e,n\}}^t = \{n - e, n\}.$$

2 - k, n are called $\bar{f}_t^{\eta,K}$ -connected and the corresponding event is denoted by $\left\{k \xleftrightarrow{\bar{f}_t^{\eta,K}} n\right\}$, if

$$\text{they are } \bar{h}_t\text{-connected and } \mathbf{B}^t(k, n; \eta, K) = \emptyset.$$

Definition 13 Let $k, n \in \mathbb{Z}^d$ be connected. Then:

1 - k, n are called $\tilde{h}_t^{\eta, K}$ -connected and the corresponding event is denoted by $\left\{ k \xleftrightarrow{\tilde{h}_t^{\eta, K}} n \right\}$,
if:

$$1.a - \mathbf{C}_{\{k, n\}} \cap \mathcal{H}_k^{t, +} \subseteq (k - Kx_t) + \mathcal{C}_\eta(t);$$

$$1.b - \mathbf{C}_{\{k, n\}} \cap \mathcal{S}_{\{k-e, k\}}^t = \{k - e, k\}.$$

2 - k, n are called $\tilde{f}_t^{\eta, K}$ -connected and the corresponding event is denoted by $\left\{ k \xleftrightarrow{\tilde{f}_t^{\eta, K}} n \right\}$, if
they are $\tilde{h}_t^{\eta, K}$ -connected and $\mathbf{B}^t(k, n; \eta, K) = \emptyset$.

The probabilities $\mathbb{P}_p \left\{ k \xleftrightarrow{\bar{h}_t} n \right\} := \bar{h}_t(k, n)$ and $\mathbb{P}_p \left\{ k \xleftrightarrow{\tilde{h}_t^{\eta, K}} n \right\} := \tilde{h}_t^{\eta, K}(k, n)$ are translation invariant, bounded from above by $e^{-\xi_p(n-k)}$ and show an asymptotic behaviour similar to that of $h_t^{\eta, K}$ given in (43), that is there exist two analytic functions on \mathbb{S}^{d-1} , $\bar{\Lambda}_p(\cdot, t)$ and $\tilde{\Lambda}_p(\cdot, t)$ such that, for x in a neighbourhood of x_t ,

$$\bar{h}_t([Nx]) = \frac{\bar{\Lambda}_p\left(\frac{x}{\|x\|}, t\right)}{\sqrt{2\pi N^{d-1} \|x\|^{d-1}}} e^{-\xi_p([Nx])} (1 + o(1)), \quad (44)$$

$$\tilde{h}_t^{\eta, K}([Nx]) = \frac{\tilde{\Lambda}_p\left(\frac{x}{\|x\|}, t\right)}{\sqrt{2\pi N^{d-1} \|x\|^{d-1}}} e^{-\xi_p([Nx])} (1 + o(1)). \quad (45)$$

Denoting by $d_t^{\eta, K}(k, n)$ the probability of the event $\{k \longleftrightarrow n, \mathbf{B}^t(k, n; \eta, K) = \emptyset\}$, which is also translation invariant, we obtain

$$\mathbb{P}_p\{0 \longleftrightarrow n\} = d_t^{\eta, K}(n) + \sum_{b_1, b_2 \in \mathbb{Z}^d} \bar{f}_t^{\eta, K}(b_2) h_t^{\eta, K}(b_1 - b_2) \tilde{f}_t^{\eta, K}(n - b_1). \quad (46)$$

2.3 Renormalization

We now follow [CI] subsection 2.2 and [CIV] section 2.

Let us represent a self-avoiding open path γ connecting the points $k, n \in \mathbb{Z}^d$ by the sequence of points $(n, i_1, \dots, i_{n-1}, k)$. Given $\eta \in (0, 1)$ and a sufficiently large renormalization scale $M > 0$, let $\gamma_M = \{n = x_1, \dots, x_{m(k)} = k\}$ be the M -skeleton of γ ([CIV] section 2.2).

If, for any $t \in \partial \mathbf{K}^p$.

$$S_t(x) := \xi_p(x) - (t, x) \quad (47)$$

denotes the *surchage function* in the direction of t , then

$$\mathcal{C}_\eta(t) = \{x \in \mathbb{R}^d : S_t(x) \leq \eta \xi_p(x)\}. \quad (48)$$

Let us define

$$\mathbb{B}_\eta^t(\gamma_M) := \{2 \leq l \leq m(k) : x_{l-1} - x_l \notin \mathcal{C}_\eta(t)\}, \quad (49)$$

where, for $l \in \mathbb{B}_\eta^t(\gamma_M)$, $x_{l-1} - x_l$ are the increments of the path γ_M backtracking with respect to $\mathcal{C}_\eta(t)$.

Notice that, by (47), if $l \in \mathbb{B}_\eta^t(\gamma_M)$, then

$$S_t(x_{l-1} - x_l) \geq \eta M. \quad (50)$$

Definition 14 We call $x_i \in \gamma_M$, $i = 2, \dots, m(k)$, a (t, η) -good point of γ_M , if

$$\gamma_M \cap (x_i + \mathcal{C}_\eta(t)) = \{x_i, \dots, x_1\} \quad (51)$$

and denote by $\mathcal{G}_\eta^t(\gamma_M)$ the set of (t, η) -good points of γ_M .

We remark that $\mathcal{G}_\eta^t(\gamma_M)$ is a totally ordered set with respect to the scalar product with t and choose the same ordering of the set of the (η, K, t) -break points given in Definition 9 that is, given $x_j, x_l \in \mathcal{G}_\eta^t(\gamma_M)$, $x_j > x_l$ if $(x_j - n, t) < (x_l - n, t)$.

Definition 15 Let us set

$$\mathcal{B}_\eta^t(\gamma_M) := \bigvee_{i \geq 1} \{l_i, \dots, r_i - 1\}, \quad (52)$$

where

$$l_1 := \max\{j \geq 1 : x_j \notin \mathcal{G}_\eta^t(\gamma_M)\}, \quad (53)$$

$$r_1 := \max\{1 \leq j < l_1 : x_j - x_{l_1} \notin \mathcal{C}_\eta(t)\}, \quad (54)$$

$$l_i := \max\{1 \leq j \leq r_{i-1} : x_j \notin \mathcal{G}_\eta^t(\gamma_M)\}, \quad (55)$$

$$r_i := \max\{1 \leq j < l_i : x_j - x_{l_i} \notin \mathcal{C}_\eta(t)\} \quad (56)$$

and denote by

$$\mathbf{x}(\mathcal{B}_\eta^t(\gamma_M)) := \{x_j \in \gamma_M : j \in \mathcal{B}_\eta^t(\gamma_M)\}, \quad (57)$$

the set of (t, η) -bad points of γ_M .

We remark that, proceeding as in the proof of [CIV] Lemma 2.2, for any $\gamma_M = \{x_1, \dots, x_m\}$, by (50) we obtain

$$\sum_k \sum_{j=l_k+1}^{r_k} S_t(x_{j-1} - x_j) \geq c_6 \eta M |\mathcal{B}_\eta^t(\gamma_M)|, \quad (58)$$

with c_6 a positive constant.

Let $\mathbf{n} \in \mathbb{Z}^{3d}$, $\mathbf{C}_\mathbf{n} := \mathbf{C}_{\{n_1, n_2, n_3\}}$ and k be connected to n_1, n_2, n_3 . Then, following [CI] subsection 2.4, we introduce the M -tree skeleton $\Gamma_\mathbf{n}^M$ of $\mathbf{C}_\mathbf{n} = \mathbf{C}_{\{k, n_1, n_2, n_3\}}$, such that

$$\Gamma_\mathbf{n}^M := \bigcup_{i=1,2,3} \gamma_M^i \bigvee L_M, \quad (59)$$

where:

- for $i = 1, 2, 3$, γ_M^i is the *self-avoiding trunk* of $\Gamma_\mathbf{n}^M$ in the direction t_i , dual to $n_i - k$, defined as in subsection 2.4 of [CI]. On the other hand, if there exist three disjoint self-avoiding open paths $\gamma_1, \gamma_2, \gamma_3$, connecting k to n_1, n_2, n_3 , we can always choose the self-avoiding trunks $\gamma_M^1, \gamma_M^2, \gamma_M^3$ to be the M -skeletons of these paths. In this case, by construction, $\cap_{i=1,2,3} \gamma_M^i = \{k\}$.
- L_M is the set of *leaves* of $\Gamma_\mathbf{n}^M$, i.e. the set of those points of $\Gamma_\mathbf{n}^M$ which do not belong to any of the self-avoiding trunks $\gamma_M^1, \gamma_M^2, \gamma_M^3$, defined by means of the construction given below.

Let us set $\mathbf{C}_\mathbf{n}^M := \bigcup_{y \in \Gamma_\mathbf{n}^M} M\mathbf{U}^p(y)$ and, for $i = 1, 2, 3$, $\mathbf{C}_{\{k, n_i\}}^M := \bigcup_{y \in \Gamma_i^M} M\mathbf{U}^p(y)$, where $\Gamma_i^M := \gamma_M^i \bigvee L_M^i$ with L_M^i the set of leaves attached to the trunk γ_M^i .

We say that $\mathbf{C}_\mathbf{n}$ is compatible with $\Gamma_\mathbf{n}^M$, and denote this fact by $\mathbf{C}_\mathbf{n} \sim \Gamma_\mathbf{n}^M$, if $\Gamma_\mathbf{n}^M$ is the M -tree skeleton of $\mathbf{C}_\mathbf{n}$, that is, if for any $m \in \mathbf{C}_\mathbf{n}$, there exists $y \in \Gamma_\mathbf{n}^M$ such that $m \in M\mathbf{U}^p(y)$. Furthermore, since $\Gamma_\mathbf{n}^M = \bigcup_{i=1,2,3} \Gamma_i^M$, from the compatibility relation $\mathbf{C}_\mathbf{n} \sim \Gamma_\mathbf{n}^M$ follows the compatibility relation $\mathbf{C}_\mathbf{n} \sim \Gamma_i^M$, $i = 1, 2, 3$.

The construction of $\Gamma_\mathbf{n}^M$ is similar the one described in section 2.4 of [CI] and can be carried out algorithmically.

step 0 Define $\Gamma_\mathbf{n}^M = \bigcup_{i=1,2,3} \gamma_M^i$ and accordingly $\mathbf{C}_\mathbf{n}^M$. Set $i := 1$.

step 1 Define $\Gamma_i^M = \gamma_M^i = \{x_1 = n_i, \dots, x_{m_i} = k\}$ and accordingly $\mathbf{C}_{\{k, n_i\}}^M$.

- If $\forall y \in \mathbf{C}_\mathbf{n} \setminus \left((\mathbf{C}_\mathbf{n}^M \setminus \mathbf{C}_{\{k, n_i\}}^M) \cap \mathbf{C}_\mathbf{n} \right)$, $\min_{z \in \mathbf{C}_{\{k, n_i\}}^M} \xi_p(y - z) \leq M$, then go to step $l_i + 1$.

- Otherwise, proceed to the following update step.

step 2 (*update step*) Reorder the points of $\Gamma_i^M = \{y_1, \dots, y_{l_i}\}$ according to lexicographical order starting from $y_1 = n_i$. Denoting by l_i the cardinality of Γ_i^M , set $j := 1$.

step j ($j \leq l_i$) Screen the lattice points $y \notin M(\mathbf{U}^p(y_j) \setminus \partial \mathbf{U}^p(y_j))$ which are endpoints of the edges crossing $M\partial \mathbf{U}^p(y_j)$ in the lexicographical order and denote their collection by $M\bar{\partial} \mathbf{U}^p(y_j)$.

- If there exists $y \in M\bar{\partial} \mathbf{U}^p(y_j)$ such that one can find a self-avoiding open path γ_y leading from y to $M\partial \mathbf{U}^p(y)$ inside $\mathbb{Z}^d \setminus \mathbf{C}_{\mathbf{n}}^M$, then set

$$L_M^i := L_M^i \cup \{y\}, \quad \Gamma_i^M := (L_M^i \cup \{y\}) \vee \gamma_M^i, \quad (60)$$

$$\mathbf{C}_{\{k, n_i\}}^M := \mathbf{C}_{\{k, n_i\}}^M \cup M\mathbf{U}^p(y), \quad \mathbf{C}_{\mathbf{n}}^M := \mathbf{C}_{\mathbf{n}}^M \cup M\mathbf{U}^p(y) \quad (61)$$

and go back to the update step.

- Otherwise, set $j := j + 1$ and go to step j .

step $l_i + 1$ Set $i := i + 1$.

- If $i = 4$, then stop.
- Otherwise, go to step 1.

By construction, $L_M = \bigvee_{i=1,2,3} L_M^i$.

We now define, for $R \in \mathbb{N}$ sufficiently large,

$$j_0 := \min \{j \geq 1 : x_j \in \mathcal{G}_\eta^{t_i}(\gamma_M^i)\} \quad (62)$$

$$j_{l+1} := \min \{j \geq j_l : \|x_j - x_{j_l}\| \geq RM, x_j \in \mathcal{G}_\eta^{t_i}(\gamma_M^i)\} \quad l \geq 0 \quad (63)$$

and consequently

$$L_i^{M \text{ bad}} := \left\{ y \in L_M^i : y \notin \bigcup_{l \geq 1} \left\{ \{RM\mathbf{U}^p(x_{j_l}) + \mathcal{C}_\eta(t_i)\} \cap \mathcal{S}_{\{x_{j_l}, x_{j_{l-1}}\}}^{t_i} \right\} \right\} \quad (64)$$

$$\Gamma_i^{M \text{ bad}} := L_i^{M \text{ bad}} \bigvee \mathbf{x}(\mathcal{B}_\eta^{t_i}(\gamma_M^i)), \quad \Gamma_i^{M \text{ good}} = \Gamma_i^M \setminus \Gamma_i^{M \text{ bad}}, \quad (65)$$

$$\mathbf{C}_{\{k, n_i\}}^{M \text{ bad}} := \bigcup_{y \in \Gamma_i^{M \text{ bad}}} M\mathbf{U}^p(y), \quad \mathbf{C}_{\{k, n_i\}}^{M \text{ good}} = \bigcup_{y \in \Gamma_i^{M \text{ good}}} M\mathbf{U}^p(y). \quad (66)$$

Moreover,

$$\Gamma_{\mathbf{n}}^{M \text{ bad}} := \bigcup_{i=1,2,3} \Gamma_i^{M \text{ bad}}, \quad \mathbf{C}_{\mathbf{n}}^{M \text{ bad}} := \bigcup_{i=1,2,3} \mathbf{C}_{\{k, n_i\}}^{M \text{ bad}} \subset \mathbf{C}_{\mathbf{n}}^M. \quad (67)$$

Hence, $|\Gamma_i^{M\text{ bad}}| = |L_i^{M\text{ bad}}| + |\mathcal{B}_\eta^{t_i}(\gamma_M^i)|$ and $|\Gamma_{\mathbf{n}}^{M\text{ bad}}| \leq \sum_{i=1}^3 |\Gamma_i^{M\text{ bad}}|$. Proceeding as in [CI] Lemma 2.3, it is possible to prove that, for any $\delta > 0$, there exists a positive constant c_7 such that, for any $i = 1, 2, 3$,

$$\mathbb{P}_p \left[\{k \longleftrightarrow n_i\} \cap \left\{ |L_i^{M\text{ bad}}| \geq \frac{\delta}{M} \|n_i - k\| \right\} \right] \leq e^{-\xi_p(n_i - k) - c_7 \delta \|n_i - k\|}, \quad (68)$$

Moreover, by (47) and (58), arguing as in [CI] Lemma 2.2, we have that, for values of M large enough, there exists a positive constant c_8 such that, for any $i = 1, 2, 3$,

$$\mathbb{P}_p \left[\{\gamma_M^i : k \xleftrightarrow{\gamma_M^i} n_i\} \cap \left\{ |\mathcal{B}_\eta^{t_i}(\gamma_M^i)| \geq \frac{\delta}{M} \|n_i - k\| \right\} \right] \leq e^{-c_8 \delta \eta \|n_i - k\| - \xi_p(n_i - k)}. \quad (69)$$

Definition 16 For any $\delta > 0$, an M -tree skeleton $\Gamma_{\mathbf{n}}^M$ is δ -good if, for any $i = 1, 2, 3$:

1. $|L_i^{M\text{ bad}}| \leq \frac{\delta}{M} \|n_i - k\|$;
2. $|\mathcal{B}_\eta^{t_i}(\gamma_M^i)| \leq \frac{\delta}{M} \|n_i - k\|$.

Let us define the slabs

$$\mathcal{S}_{l,i}^{M,R} := \mathcal{S}_{\{k+l4RM \frac{t_i}{\|t_i\|}, k+(l+1)4RM \frac{t_i}{\|t_i\|}\}}^{t_i} \quad i = 1, 2, 3; l \in \mathbb{N}. \quad (70)$$

For any $i = 1, 2, 3$, $\mathbf{C}_{\mathbf{n}}^M$ intersects \mathcal{N} subsequent $\mathcal{S}_{l,i}^{M,R}$ slabs, with $\mathcal{N} \geq \frac{(1-\eta)c_-(p)}{4RM} \|n_i - k\|$. Furthermore, if $\Gamma_{\mathbf{n}}^M$ is δ -good, at most $\frac{2\delta\|n_i - k\|}{M}$ of the $\mathcal{S}_{l,i}^{M,R}$ slabs contain points belonging to $\mathbf{C}_{\{k,n_i\}}^{M\text{ bad}}$. Hence, if we choose $\delta \in \left(0, \frac{(1-\eta)c_-(p)}{16R}\right)$, the number of $\mathcal{S}_{l,i}^{M,R}$ slabs containing only points of $\mathbf{C}_{\{k,n_i\}}^{M\text{ good}}$, which we will call δ -good slabs, is larger than $\frac{1}{2} \frac{(1-\eta)c_-(p)}{4RM} \|n_i - k\|$. Renumbering all the δ -good $\mathcal{S}_{l,i}^{M,R}$ slabs as $\mathcal{S}_{l_1,i}^{M,R}, \dots, \mathcal{S}_{l_r,i}^{M,R}$, $r \geq \frac{(1-\eta)c_-(p)}{8RM} \|n_i - k\|$, for every δ -good $\mathcal{S}_{l_j,i}^{M,R}$ slab and every cluster $\mathbf{C}_{\mathbf{n}}$ compatible with $\Gamma_{\mathbf{n}}$, we have

$$\mathbf{C}_{\mathbf{n}} \cap \mathcal{S}_{l_j,i}^{M,R} \subseteq \mathcal{R}_{l_j,i}^{M,R} := \bigcup_{x \in \Gamma_i^M \cap \mathcal{S}_{l_j,i}^{M,R}} 4M\mathbf{U}^p(x). \quad (71)$$

Choosing K such that $\frac{K}{M} > 1$ and setting

$$\text{dist} \left(\mathcal{R}_{l_j,i}^{M,R}; \mathcal{R}_{l_{j'},i}^{M,R} \right) := \min_{\substack{y \in \mathcal{R}_{l_j,i}^{M,R} \\ y' \in \mathcal{R}_{l_{j'},i}^{M,R}}} \xi_p(y - y'), \quad (72)$$

we define the application $\{1, \dots, r\} \ni j \mapsto q(j) = j_q \in \{1, \dots, s\}$ such that

$$j_1 := 1, \quad (73)$$

$$j_{q+1} := \min \left\{ j > j_q : \text{dist} \left(\mathcal{R}_{l_j, i}^{M, R}, \mathcal{R}_{l_{j_q}, i}^{M, R} \right) \geq \frac{2K}{\eta} \right\}. \quad (74)$$

As in [CI] section 3.3 and 4, for each $i = 1, 2, 3$, it is possible to modify at most $c_9 (RM)^d$ bonds inside the regions $\mathcal{R}_{l_{j_1}, i}^{M, R}, \dots, \mathcal{R}_{l_{j_s}, i}^{M, R}$, $s \geq \frac{\eta(1-\eta)c_-(p)}{16KRM} \|n_i - k\|$, in an independent way such that the resulting modified cluster is still compatible with Γ_i^M and contains at least one t_i -break point located in each these regions. By construction, these t_i -break points verify condition 2 of Definition 9. Since for any $x \in \mathcal{C}_\eta(t)$,

$$\|P_i^\perp x\| \leq \sqrt{\frac{1 - (1-\eta)^2 c_-^2(p)}{(1-\eta)^2 c_-^2(p)}} (t_i, x), \quad (75)$$

we can choose $\frac{K}{M} > 4\eta R$ large enough such that at least half the points in $\mathcal{R}_{l_{j_{q+1}}, i}^{M, R}$ belong to $z + \mathcal{C}_\eta(t_i)$, for every $z \in \mathcal{R}_{l_{j_q}, i}^{M, R}$. Therefore, at least half of the t_i -break points of the modified cluster satisfy condition 1 of Definition 9. We also remark that, since by construction any t_i -break point of the modified cluster belongs to a neighborhood $M\mathbf{U}^p(x_j)$ of some good point x_j , one can choose $R > \frac{(1-\eta)c_-(p)}{2\sqrt{1-(1-\eta)^2 c_-^2(p)}}$ large enough such that, for any $q = 1, \dots, s$, there are at least one t_i -break point $b_q \in \mathcal{R}_{l_{j_q}, i}^{M, R}$ and one t_i -break point $b_{q+1} \in \mathcal{R}_{l_{j_{q+1}}, i}^{M, R}$ which verify conditions 3 of Definition 9, provided that the slabs $\mathcal{S}_{l_j, i}^{M, R}$, for $l_{j_q} < l_j < l_{j_{q+1}}$, are δ -good. On the other hand, if $\mathcal{S}_{l_j, i}^{M, R}$ contains points of $L_i^{M, bad}$, at most only the t_i -break points of the modified cluster belonging to $\mathcal{S}_{l_{j_{q+1}}, i}^{M, R}, \dots, \mathcal{S}_{l_{j_q+\rho}, i}^{M, R}$, with $\rho \leq [\frac{1}{4}s]$, do not verify condition 3 of 9.

Therefore, proceeding as in the proofs of Lemma 4.1 in [CI], this argument, together with the estimates (68) (69), proves that for any $\eta > 0$ sufficiently small, there exists $\delta_1 = \delta_1(\eta, p)$ and $c_{10} = c_{10}(\eta, p) > 0$ such that

$$\mathbb{P}_p \left[\left\{ |\mathbf{B}^{t_i}(k, n_i; \eta, K)| < \delta_1 \|n_i - k\| \right\} \cap \left\{ k \xleftrightarrow{\bar{h}_{t_i}} n_i \right\} \right] \leq e^{-\xi_p(n_i - k) - c_{10}\|n_i - k\|}. \quad (76)$$

Since \bar{h}_{t_i} and $\bar{f}_{t_i}^{\eta, K}$ satisfy a renewal equation analogous to (42), the last inequality implies that $\bar{f}_{t_i}^{\eta, K}$ verifies a *mass-gap* type condition similar to the one verified by f_{t_i} ([CI] section 4), that is there exists a positive constant $c_{11} = c_{11}(\eta, p)$ such that

$$\frac{\bar{f}_{t_i}^{\eta, K}(k - n_i)}{\bar{h}_{t_i}(k - n_i)} \leq e^{-c_{11}\|n_i - k\|}. \quad (77)$$

Hence, since $f_{t_i}^{\eta,K}(n_i - k) < \bar{f}_{t_i}^{\eta,K}(n_i - k)$, a similar estimate holds also for $f_{t_i}^{\eta,K}$.

The BK inequality and the previous construction also imply

$$\mathbb{P}_p[F(k; \mathbf{n}) \cap \{|\mathbf{B}^{t_i}(k, n_i; \eta, K)| < \delta_1 \|n_i - k\|\}] \leq e^{-\varphi_{p,\mathbf{n}}(k) - c_{10}\|n_i - k\|} \quad i = 1, 2, 3. \quad (78)$$

Let us now consider the event

$$G_{\delta_2}^{\eta,K}(k; \mathbf{n}) := F(k; \mathbf{n}) \cap \bigcap_{i=1,2,3} \left\{ k \overset{\bar{h}_{t_i}}{\longleftrightarrow} n_i; |\mathbf{B}^{t_i}(k, n_i; \eta, K)| \leq \delta_2 \|n_i - k\| \right\} \quad (79)$$

with $\delta_2 \leq \delta_1$. To estimate the probability of $G_{\delta_2}^{\eta,K}(k; \mathbf{n})$ we can repeat the same renormalization procedure previously set up to prove the mass-gap type condition for the $f_{t_i}^{\eta,K}$ connections along one direction $\frac{t_i}{\|t_i\|}$, except that now we need to consider all the three directions $\frac{t_1}{\|t_1\|}, \frac{t_2}{\|t_2\|}, \frac{t_3}{\|t_3\|}$, at once. Given M -tree skeleton $\Gamma_{\mathbf{n}}^M$, by the BK inequality it follows that $\mathbb{P}_p[\Gamma_{\mathbf{n}}^M] \leq \prod_{i=1}^3 \mathbb{P}_p[\Gamma_i^M]$. Consequently we obtain

$$\mathbb{P}_p[G_{\delta_2}^{\eta,K}(k; \mathbf{n})] \leq e^{-\varphi_{p,\mathbf{n}}(k) - c_{12} \sum_{i=1}^3 \|n_i - k\|}. \quad (80)$$

2.4 Proof of Theorem 1

Definition 17 Given $\eta \in (0, 1)$ and K sufficiently large, let $\mathbf{n} \in X'_3$ and let $\mathbf{t} = (t_1, t_2, t_3)$ be the vector in $(\mathbb{R}^d)^3$ whose entries, t_1, t_2, t_3 are respectively the polar points to $n_1 - x_0(\mathbf{n}), n_2 - x_0(\mathbf{n}), n_3 - x_0(\mathbf{n})$. By (34), for any $k \in \mathbb{Z}^d$ and $i = 1, 2, 3$, we denote by b_{i,μ_i} the element of $\mathbf{B}^{t_i}(k, n_i; \eta, K)$ such that the scalar product $|(n_i - b_{i,\mu_i}, t_i)|$ is maximal and define $T(\mathbf{b}; k, \mathbf{n}) = T(b_1, b_2, b_3; k, \mathbf{n})$ to be the event that k is connected to n_1, n_2, n_3 by three self-avoiding disjoint open paths incidents in b_1, b_2, b_3 , these being the positions assumed respectively by the random points $b_{1,\mu_1-1}, b_{2,\mu_2-1}, b_{3,\mu_3-1}$. Moreover, any configuration $\mathbf{b} \in (\mathbb{Z}^d)^3$ for the (η, K, \mathbf{t}) -break points $b_{1,\mu_1-1}, b_{2,\mu_2-1}, b_{3,\mu_3-1}$ will be called admissible for k if $\mathbb{P}_p[T(\mathbf{b}; k, \mathbf{n})] > 0$.

We remark that, given $\mathbf{n} \in X'_3$ and any $k_1, k_2 \in \mathbb{Z}^d$, if we choose two distinct vectors $\mathbf{b}_1, \mathbf{b}_2 \in (\mathbb{Z}^d)^3$, then $T(\mathbf{b}_1; k_1, \mathbf{n})$ and $T(\mathbf{b}_2; k_2, \mathbf{n})$ are disjoint.

For any $\mathbf{b} \in (\mathbb{Z}^d)^3$, let $\mathcal{T}(\mathbf{b}) := \bigcap_{i=1}^3 \mathcal{H}_{b_i}^{t_i, -}$. Then, from the previous definition, it follows that \mathbf{b} cannot be admissible for $k \in \mathbb{Z}^d$ if $k \notin \mathcal{T}(\mathbf{b})$. If $k_1, k_2 \in \mathbb{Z}^d$ and \mathbf{b} is admissible for both k_1 and k_2 , then $T(\mathbf{b}; k_1, \mathbf{n})$ and $T(\mathbf{b}; k_2, \mathbf{n})$ need not be disjoint.

Therefore, the event $F(k; \mathbf{n})$ allows the decomposition

$$F(k; \mathbf{n}) = \bigvee_{b_1, b_2, b_3 \in \mathbb{Z}^d} T(b_1, b_2, b_3; k, \mathbf{n}) \bigvee T^*(k; \mathbf{n}), \quad (81)$$

with $T^*(k; \mathbf{n}) = F(k; \mathbf{n}) \cap \bigcup_{i=1,2,3} \{|\mathbf{B}^{t_i}(k, n_i; \eta, K)| \leq 1\}$.

Before entering into details, let us describe the main ideas of the proof of Theorem (1).

As a first step, in the following proposition, we derive the asymptotic behaviour for the probability of the event $F\left([Nx_0(\mathbf{x}) + \sqrt{N}y]; [N\mathbf{x}]\right)$ that the point $[Nx_0(\mathbf{x}) + \sqrt{N}y]$ is connected by three disjoint self-avoiding open paths to the points $[Nx_1], [Nx_2], [Nx_3]$, as N goes to infinity. The event $F\left([Nx_0(\mathbf{x}) + \sqrt{N}y]; [N\mathbf{x}]\right)$, apart from terms that can be neglected, can be decomposed into a partition according to the positions of the points b_1, b_2, b_3 , as shown in (81), where the distances of b_i 's from $[Nx_0(\mathbf{x}) + \sqrt{N}y]$ can be assumed to be smaller than N^β with $0 < \beta < \frac{1}{2}$. The definition of the b_i 's implies that the probability of a term of such a decomposition is of the form

$$g_p^{\eta, K}(k; \mathbf{b}) \prod_{i=1,2,3} \tilde{h}_{t_i}^{\eta, K}([Nx_i] - b_i), \quad (82)$$

with $k = [Nx_0(\mathbf{x}) + \sqrt{N}y]$, where $g_p^{\eta, K}(k; \mathbf{b})$ is a translationally invariant function. The desired asymptotics follows then from the Ornstein-Zernike estimate (45) for $\tilde{h}_t^{\eta, K}$ and the expansion of the function ξ_p that appears in it.

The second step is to obtain the asymptotic behaviour of the probability of the event $E([N\mathbf{x}])$ that appears in the denominator of the conditional expectation (14) as N tends to infinity. As before, apart from terms that can be neglected, we can decompose this event into a partition according to the positions of the points b_1, b_2, b_3 . The probability of a term of such a decomposition can be written as

$$g_p^{\eta, K}(\mathbf{b}) \prod_{i=1,2,3} \tilde{h}_{t_i}^{\eta, K}([Nx_i] - b_i), \quad (83)$$

where $g_p^{\eta, K}(\mathbf{b})$ is translation invariant. It can be assumed, neglecting terms of small probability, that the distances among the b_i 's are smaller than N^β , with $\beta \in (0, \frac{1}{2})$, and that the distance of $x_0([N\mathbf{x}])$ from each of the b_i 's are smaller than N^α , with $\alpha \in (\frac{1}{2}, 1)$. The Ornstein-Zernike estimate (45) for $\tilde{h}_t^{\eta, K}$ and the expansion of the function ξ_p that appears in it give then the desired asymptotics.

Proposition 18 *For $\mathbf{x} \in X'_3$, let $x_0 = x_0(\mathbf{x})$ be the unique minimizer of $\varphi_{p, \mathbf{x}}$. Then, for any $y \in \mathbb{R}^d$, there exists a positive real analytic function Θ_p on X'_3 such that, for $d \geq 2$ and $p < p_c(d)$,*

$$\mathbb{P}_p \left[F\left([Nx_0(\mathbf{x}) + \sqrt{N}y]; [N\mathbf{x}]\right) \right] = \frac{\Theta_p(\mathbf{x})}{(2\pi N^{d-1})^{\frac{3}{2}}} e^{-\varphi_{p, [N\mathbf{x}]}(x_0([N\mathbf{x}]) - \frac{(y, H_\varphi(x_0(\mathbf{x}), \mathbf{x}; p)y)}{2})} (1 + o(1)). \quad (84)$$

Proof. Let $k \in \mathbb{Z}^d$ be such that $\|k - x_0([N\mathbf{x}])\| \leq c_{13}N^{\frac{1}{2}}$ and denote by $t_i = t_i(\mathbf{x})$ the polar point to $x_0(\mathbf{x}) - x_i$, $i = 1, 2, 3$. We can choose $\eta \in (0, 1)$ small enough such that $\mathcal{C}_{2\eta}(t_i) \cap \mathcal{C}_{2\eta}(t_j) = \emptyset$, $i \neq j = 1, 2, 3$, then, from Lemma 11, it follows that for $i = 1, 2, 3$, $\mathbf{C}_{\{b_i, [Nx_i]\}} \cap \mathcal{H}_{b_i, \mu_i}^{t_i, +} \subset b_{i, \mu_i} + \mathcal{C}_{2\eta}(t_i)$ and for $i \neq j$

$$(b_{i, \mu_i} + \mathcal{C}_{2\eta}(t_i)) \cap (b_{j, \mu_j} + \mathcal{C}_{2\eta}(t_j)) = \emptyset. \quad (85)$$

If, for any $\mathbf{b} \in (\mathbb{Z}^d)^3$ and K sufficiently large, we define the function

$$g_p^{\eta, K}(k; \mathbf{b}) = g_p^{\eta, K}(k; b_1, b_2, b_3) := \mathbb{P}_p[G^{\eta, K}(k; b_1, b_2, b_3)], \quad (86)$$

which is the probability of the event

$$G^{\eta, K}(k; b_1, b_2, b_3) = G^{\eta, K}(k; \mathbf{b}) := F(k; \mathbf{b}) \cap \bigcap_{i=1}^3 \left\{ k \longleftrightarrow b_i, |\mathbf{B}^{t_i}(k, b_i; \eta, K)| = 1 \right\}, \quad (87)$$

then, by Definition 17, we have

$$\mathbb{P}_p[T(k; \mathbf{b}, [N\mathbf{x}])] := g_p^{\eta, K}(k; \mathbf{b}) \prod_{i=1,2,3} \tilde{h}_{t_i}^{\eta, K}([Nx_i] - b_i). \quad (88)$$

Notice that $g_p^{\eta, K}(k; \mathbf{b})$ is translation invariant, i.e.

$$g_p^{\eta, K}(k + u; b_1 + u, b_2 + u, b_3 + u) = g_p^{\eta, K}(k; b_1, b_2, b_3), \quad u \in \mathbb{Z}^d. \quad (89)$$

By (81), we have

$$\mathbb{P}_p[F(k; [N\mathbf{x}])] = \sum_{b_1, b_2, b_3 \in \mathbb{Z}^d} \mathbb{P}_p[T(b_1, b_2, b_3; k, [N\mathbf{x}])] + \mathbb{P}_p[T^*(k; [N\mathbf{x}])]. \quad (90)$$

Since by (78),

$$\mathbb{P}_p[T^*(k; [N\mathbf{x}])] \leq \sum_{i=1,2,3} e^{-\varphi_p([N\mathbf{x}](k) - c_{10}\|[Nx_i] - k\|)}, \quad (91)$$

then, by (86), we need to estimate

$$\begin{aligned} & \sum_{b_1, b_2, b_3 \in \mathbb{Z}^d} \mathbb{P}_p[T(b_1, b_2, b_3; k, [N\mathbf{x}])] \\ &= \sum_{b_1, b_2, b_3 \in \mathbb{Z}^d} g_p^{\eta, K}(k; b_1, b_2, b_3) \prod_{i=1,2,3} \tilde{h}_{t_i}^{\eta, K}([Nx_i] - b_i) \\ &= e^{-\varphi_p([N\mathbf{x}](k))} \sum_{b_1, b_2, b_3 \in \mathbb{Z}^d} g_p^{\eta, K}(k; b_1, b_2, b_3) \prod_{i=1,2,3} e^{\xi_p([Nx_i] - k)} \tilde{h}_{t_i}^{\eta, K}([Nx_i] - b_i). \end{aligned} \quad (92)$$

By (80), it follows that

$$g_p^{\eta,K}(k; b_1, b_2, b_3) \leq e^{-\varphi_{p,\mathbf{b}}(k) - c_{11} \sum_{i=1,2,3} \|b_i - k\|} \quad (93)$$

But, by the convexity of ξ_p , there exists a positive constant c_{14} such that, $\forall \beta \in (0, 1)$ and N large enough, by (93), (41) and (2), we have

$$\begin{aligned} & \sum_{b_1 \in \mathbb{Z}^d : \|b_1 - k\| > N^\beta} \sum_{b_2, b_3 \in \mathbb{Z}^d} g_p^{\eta,K}(k; b_1, b_2, b_3) \prod_{i=1}^3 e^{\xi_p([Nx_i] - k)} \tilde{h}_{t_i}^{\eta,K}([Nx_i] - b_i) \\ & \leq \sum_{b_1 \in \mathbb{Z}^d : \|b_1 - k\| > N^\beta} e^{-c_{11} \|b_1 - k\| + \xi_p([Nx_1] - k) - \xi_p([Nx_1] - b_1) - \xi_p(b_1 - k)} \times \\ & \times \prod_{i=1}^2 \sum_{b_i \in \mathbb{Z}^d} e^{-c_{11} \|b_i - k\| + \xi_p([Nx_i] - k) - \xi_p([Nx_i] - b_i) - \xi_p(b_i - k)} \leq e^{-c_{14} N^\beta} \end{aligned} \quad (94)$$

and analogous estimates hold for the sums over b_2 and b_3 . Thus, by (45), we are left with the estimate of

$$\begin{aligned} & \sum_{\substack{b_1 \in \mathbb{Z}^d : \|b_1 - k\| \leq N^\beta \\ b_2 \in \mathbb{Z}^d : \|b_2 - k\| \leq N^\beta \\ b_3 \in \mathbb{Z}^d : \|b_3 - k\| \leq N^\beta}} g_p^{\eta,K}(k; b_1, b_2, b_3) \prod_{i=1,2,3} e^{\xi_p([Nx_i] - k)} \tilde{h}_{t_i}^{\eta,K}([Nx_i] - b_i) \\ & = \sum_{\substack{b_1 \in \mathbb{Z}^d : \|b_1 - k\| \leq N^\beta \\ b_2 \in \mathbb{Z}^d : \|b_2 - k\| \leq N^\beta \\ b_3 \in \mathbb{Z}^d : \|b_3 - k\| \leq N^\beta}} g_p^{\eta,K}(k; b_1, b_2, b_3) \prod_{i=1,2,3} \frac{\tilde{\Lambda}_p\left(\frac{Nx_i - b_i}{\|Nx_i - b_i\|}, t_i\right)}{\sqrt{2\pi N^{d-1} \left\|x_i - \frac{b_i}{N}\right\|^{d-1}}} \times \\ & \times \exp[\xi_p([Nx_i] - k) - \xi_p([Nx_i] - b_i)] (1 + o(1)). \end{aligned} \quad (95)$$

By the convexity of ξ_p and by (93), we have

$$\begin{aligned} & \sum_{\substack{b_1 \in \mathbb{Z}^d : \|b_1 - k\| \leq N^\beta \\ b_2 \in \mathbb{Z}^d : \|b_2 - k\| \leq N^\beta \\ b_3 \in \mathbb{Z}^d : \|b_3 - k\| \leq N^\beta}} g_p^{\eta,K}(k; b_1, b_2, b_3) \prod_{i=1,2,3} \exp[\xi_p([Nx_i] - k) - \xi_p([Nx_i] - b_i)] \\ & \leq \left(\sum_{b_1 \in \mathbb{Z}^d : \|b_1 - k\| \leq N^\beta} e^{-c_{12} \|b_1 - k\|} \right)^3. \end{aligned} \quad (96)$$

Moreover, by translation invariance, setting $\forall i = 1, 2, 3$, $b_i = a_i + k$, for $\beta \in (0, \frac{1}{2})$,

$$\begin{aligned}
& \sum_{\substack{b_1 \in \mathbb{Z}^d : \|b_1 - k\| \leq N^\beta \\ b_2 \in \mathbb{Z}^d : \|b_2 - k\| \leq N^\beta \\ b_3 \in \mathbb{Z}^d : \|b_3 - k\| \leq N^\beta}} g_p^{\eta, K}(k; b_1, b_2, b_3) \prod_{i=1,2,3} \frac{\tilde{\Lambda}_p\left(\frac{Nx_i - b_i}{\|Nx_i - b_i\|}, t_i\right)}{\sqrt{2\pi N^{d-1} \left\|x_i - \frac{b_i}{N}\right\|^{d-1}}} \times \\
& \times \exp[\xi_p([Nx_i] - k) - \xi_p([Nx_i] - b_i)] (1 + o(1)) \\
& = \prod_{i=1,2,3} \frac{\tilde{\Lambda}_p\left(\frac{x_i - x_0(\mathbf{x})}{\|x_i - x_0(\mathbf{x})\|}, t_i\right)}{\sqrt{2\pi N^{d-1} \|x_i - x_0(\mathbf{x})\|^{d-1}}} \sum_{\substack{a_1 \in \mathbb{Z}^d : \|a_1\| \leq N^\beta \\ a_2 \in \mathbb{Z}^d : \|a_2\| \leq N^\beta \\ a_3 \in \mathbb{Z}^d : \|a_3\| \leq N^\beta}} g_p^{\eta, K}(0; a_1, a_2, a_3) \times \\
& \times \exp[\xi_p([Nx_i] - k) - \xi_p([Nx_i] - a_i - k)] (1 + o(1)).
\end{aligned} \tag{97}$$

But

$$\begin{aligned}
\xi_p([Nx_i] - k) &= \xi_p([Nx_i] - x_0([N\mathbf{x}])) - (\nabla \xi_p([Nx_i] - x_0([N\mathbf{x}])), k - x_0([N\mathbf{x}])) + \\
& + \frac{1}{2} (k - x_0([N\mathbf{x}]), H_\xi([Nx_i] - x_0([N\mathbf{x}]); p) (k - x_0([N\mathbf{x}])) + O\left(\frac{1}{\sqrt{N}}\right) \\
\xi_p([Nx_i] - a_i - k) &= \xi_p([Nx_i] - x_0([N\mathbf{x}])) - (\nabla \xi_p([Nx_i] - x_0([N\mathbf{x}])), a_i + k - x_0([N\mathbf{x}])) + \\
& + \frac{1}{2} (a_i + k - x_0([N\mathbf{x}]), H_\xi([Nx_i] - x_0([N\mathbf{x}]); p) (a_i + k - x_0([N\mathbf{x}])) + O\left(\frac{1}{\sqrt{N}}\right)
\end{aligned} \tag{98}$$

Hence, since $H_\xi(\cdot; p)$ is a homogeneous function of order -1 in $\mathbb{R}^d \setminus \{0\}$, (97) is equal to

$$\begin{aligned}
& \prod_{i=1,2,3} \frac{\tilde{\Lambda}_p\left(\frac{x_i - x_0(\mathbf{x})}{\|x_i - x_0(\mathbf{x})\|}, t_i\right)}{\sqrt{2\pi N^{d-1} \|x_i - x_0(\mathbf{x})\|^{d-1}}} \sum_{\substack{a_1 \in \mathbb{Z}^d : \|a_1\| \leq N^\beta \\ a_2 \in \mathbb{Z}^d : \|a_2\| \leq N^\beta \\ a_3 \in \mathbb{Z}^d : \|a_3\| \leq N^\beta}} g_p^{\eta, K}(0; a_1, a_2, a_3) \times \\
& \times \exp\left[\sum_{i=1,2,3} (\nabla \xi_p([Nx_i] - x_0([N\mathbf{x}])), a_i)\right] (1 + o(1)).
\end{aligned} \tag{99}$$

Since there exists a constant c_{15} such that $\|\nabla \xi_p([Nx_i] - x_0([N\mathbf{x}])) - t_i\| \leq \frac{c_{15}}{N}$, then, by (96),

$$\begin{aligned} & \sum_{\substack{a_1 \in \mathbb{Z}^d : \|a_1\| \leq N^\beta \\ a_2 \in \mathbb{Z}^d : \|a_2\| \leq N^\beta \\ a_3 \in \mathbb{Z}^d : \|a_3\| \leq N^\beta}} g_p^{\eta, K}(0; a_1, a_2, a_3) e^{\sum_{i=1,2,3} (\nabla \xi_p([Nx_i] - x_0([N\mathbf{x}]))), a_i)} \\ &= \sum_{a_1, a_2, a_3 \in \mathbb{Z}^d} g_p^{\eta, K}(0; a_1, a_2, a_3) e^{\sum_{i=1,2,3} (t_i, a_i)} (1 + o(1)) \end{aligned} \quad (100)$$

and

$$\Theta_p(\mathbf{x}) := \prod_{i=1,2,3} \frac{\tilde{\Lambda}_p\left(\frac{x_i - x_0(\mathbf{x})}{\|x_i - x_0(\mathbf{x})\|}, t_i(\mathbf{x})\right)}{\sqrt{\|x_i - x_0(\mathbf{x})\|^{d-1}}} \sum_{a_1, a_2, a_3 \in \mathbb{Z}^d} g_p^{\eta, K}(0; a_1, a_2, a_3) e^{\sum_{i=1,2,3} (t_i(\mathbf{x}), a_i)} \quad (101)$$

is an analytic function on X'_3 .

Furthermore,

$$\begin{aligned} \varphi_{p, [N\mathbf{x}]}(k) &= \varphi_{p, [N\mathbf{x}]}(x_0([N\mathbf{x}])) + \\ &+ \frac{1}{2N} \left(k - x_0([N\mathbf{x}]), H_\varphi\left(\frac{x_0([N\mathbf{x}])}{N}, \frac{[N\mathbf{x}]}{N}; p\right) (k - x_0([N\mathbf{x}])) \right) + O\left(\frac{1}{\sqrt{N}}\right). \end{aligned} \quad (102)$$

Hence, by (28), setting $k = [Nx_0(\mathbf{x}) + \sqrt{N}y]$, we obtain

$$\varphi_{p, [N\mathbf{x}]}([Nx_0(\mathbf{x}) + \sqrt{N}y]) = \varphi_{p, [N\mathbf{x}]}(x_0([N\mathbf{x}])) + \frac{1}{2} (y, H_\varphi(x_0(\mathbf{x}), \mathbf{x}; p) y) + O\left(\frac{1}{\sqrt{N}}\right). \quad (103)$$

■

By (23), for any $\alpha \in (\frac{1}{2}, 1)$, $\mathbf{x} \in (\mathbb{R}^d)^3$, we have

$$E([N\mathbf{x}]) = A_{\alpha, N}(\mathbf{x}) \bigvee A_{\alpha, N}^*(\mathbf{x}), \quad (104)$$

$$A_{\alpha, N}^*(\mathbf{x}) := \bigcup_{k \in \mathbb{Z}^d : \|k - x_0([N\mathbf{x}])\| \leq N^\alpha} F(k; [N\mathbf{x}]), \quad (105)$$

and by (81),

$$A_{\alpha, N}^*(\mathbf{x}) = \bigcup_{k \in \mathbb{Z}^d : \|k - x_0([N\mathbf{x}])\| \leq N^\alpha} \left\{ \bigvee_{\mathbf{b} \in (\mathbb{Z}^d)^3} T(\mathbf{b}; k, [N\mathbf{x}]) \bigvee T^*(k; [N\mathbf{x}]) \right\}. \quad (106)$$

Thus, because of Proposition 4, we are left with the estimate of the events

$$A_{\alpha,N}^{**}(\mathbf{x}) := \bigcup_{k \in \mathbb{Z}^d : \|k - x_0([N\mathbf{x}])\| \leq N^\alpha} \bigvee_{\mathbf{b} \in (\mathbb{Z}^d)^3} T(\mathbf{b}; k, [N\mathbf{x}]), \quad (107)$$

$$T_{\alpha,N}^*(\mathbf{x}) := \bigcup_{k \in \mathbb{Z}^d : \|k - x_0([N\mathbf{x}])\| \leq N^\alpha} T^*(k; [N\mathbf{x}]) \quad (108)$$

but, as we have already remarked, given $k_1, k_2 \in \mathbb{Z}^d$ and a $\mathbf{b} \in (\mathbb{Z}^d)^3$ admissible for both k_1 and k_2 , in general $T(\mathbf{b}; k_1, [N\mathbf{x}]) \cap T(\mathbf{b}; k_2, [N\mathbf{x}]) \neq \emptyset$. Hence we cannot use simply the asymptotic estimate (84) and sum directly over k .

Let us define, for any $\mathbf{b} \in (\mathbb{Z}^d)^3$, the event $T(\mathbf{b}; [N\mathbf{x}]) := \bigcup_{k \in \mathbb{Z}^d} T(\mathbf{b}; k, [N\mathbf{x}])$ and notice that, if $\mathbf{b}_1 \neq \mathbf{b}_2$, then $T(\mathbf{b}_1; [N\mathbf{x}])$ and $T(\mathbf{b}_2; [N\mathbf{x}])$ are disjoint.

We also remark that the probability of $T(\mathbf{b}; [N\mathbf{x}])$ depends only on the vectors $b_i - b_j$, $i \neq j = 1, 2, 3$ and therefore is translation invariant.

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. By the definition of $T(\mathbf{b}; [N\mathbf{x}])$ we have

$$E([N\mathbf{x}]) = \bigvee_{b_1, b_2, b_3 \in \mathbb{Z}^d} T(b_1, b_2, b_3; [N\mathbf{x}]) \bigvee T^*([N\mathbf{x}]), \quad (109)$$

where $T^*([N\mathbf{x}]) := \bigcup_{k \in \mathbb{Z}^d} T^*(k; [N\mathbf{x}])$. Hence

$$\mathbb{P}_p[E([N\mathbf{x}])] = \sum_{b_1, b_2, b_3 \in \mathbb{Z}^d} \mathbb{P}_p[T(b_1, b_2, b_3; [N\mathbf{x}])] + \mathbb{P}_p[T^*([N\mathbf{x}])]. \quad (110)$$

Proceeding as in the proof of the previous proposition, for any $\mathbf{b} \in (\mathbb{Z}^d)^3$, we define the function

$$g_p^{\eta,K}(\mathbf{b}) = g_p^{\eta,K}(b_1, b_2, b_3), \quad (111)$$

which is the probability of the event $G^{\eta,K}(\mathbf{b}) := \bigcup_{k \in \mathcal{T}(\mathbf{b})} G^{\eta,K}(k; \mathbf{b})$. Then

$$\mathbb{P}_p[T(\mathbf{b}, [N\mathbf{x}])] := g_p^{\eta,K}(\mathbf{b}) \prod_{i=1,2,3} \tilde{h}_{t_i}^{\eta,K}([Nx_i] - b_i). \quad (112)$$

By the translation invariance of $g_p^{\eta,K}(\mathbf{b})$, setting $b_1 = b$, $b_2 = b + a_1$, $b_3 = b + a_2$, we obtain

$$\begin{aligned} \sum_{b_1, b_2, b_3 \in \mathbb{Z}^d} \mathbb{P}_p[T(b_1, b_2, b_3; [N\mathbf{x}])] &= \sum_{b_1, b_2, b_3 \in \mathbb{Z}^d} g_p^{\eta,K}(b_1, b_2, b_3) \prod_{i=1,2,3} \tilde{h}_{t_i}^{\eta,K}([Nx_i] - b_i) \quad (113) \\ &= e^{-\varphi_p([N\mathbf{x}](x_0([N\mathbf{x}])))} \sum_{b \in \mathbb{Z}^d} \tilde{h}_{t_1}^{\eta,K}([Nx_1] - b) e^{\xi_p([Nx_1] - x_0([N\mathbf{x}])))} \times \\ &\times \sum_{a_1, a_2 \in \mathbb{Z}^d} g_p^{\eta,K}(0, a_1, a_2) \prod_{i=2,3} e^{\xi_p([Nx_i] - x_0([N\mathbf{x}])))} \tilde{h}_{t_i}^{\eta,K}([Nx_i] - b - a_{i-1}). \end{aligned}$$

Since,

$$\mathcal{T}(\mathbf{b}) = \mathcal{T}(b, b + a_1, b + a_2) = b + \mathcal{T}(0, a_1, a_2), \quad (114)$$

by (93),

$$\begin{aligned} e^{\varphi_p([N\mathbf{x}](x_0([N\mathbf{x}])))} \sum_{b \in \mathbb{Z}^d} \tilde{h}_{t_1}^{\eta,K}([Nx_1] - b) \sum_{a_1, a_2 \in \mathbb{Z}^d} g_p^{\eta,K}(0, a_1, a_2) \prod_{i=2,3} \tilde{h}_{t_i}^{\eta,K}([Nx_i] - b - a_{i-1}) \quad (115) \\ \leq \sum_{b \in \mathbb{Z}^d} \sum_{a_1, a_2 \in \mathbb{Z}^d} \sum_{k \in \mathcal{T}(0, a_1, a_2; [N\mathbf{x}]) \cap \mathbb{Z}^d} \exp \{ -c_{12} [\|k\| + \|k - a_1\| + \|k - a_2\|] + \\ -\xi_p(k) - \xi_p(k - a_1) - \xi_p(k - a_2) - \xi_p([Nx_1] - b) + \xi_p([Nx_1] - x_0([N\mathbf{x}])) \\ -\xi_p([Nx_2] - b - a_1) + \xi_p([Nx_2] - x_0([N\mathbf{x}])) - \xi_p([Nx_3] - b - a_2) + \xi_p([Nx_3] - x_0([N\mathbf{x}])) \}. \end{aligned}$$

We recall that the probability that points in $\mathcal{T}(\mathbf{b})$ disjointly connected to $[Nx_1], [Nx_2], [Nx_3]$, lie outside of a neighborhood of $x_0([N\mathbf{x}])$ of radius N^α , with $\alpha > \frac{1}{2}$, is smaller than the r.h.s. of (24). Hence, we can restrict ourselves to consider only those configurations of points b_1, b_2, b_3 , such that the associated set $\mathcal{T}(\mathbf{b})$ has non-empty intersection with $N^\alpha \mathbf{U}^p(x_0([N\mathbf{x}]))$. Making use of the shorthand notation \sum_k^I for $\sum_{k \in \mathcal{T}(0, a_1, a_2) \cap N^\alpha \mathbf{U}^p(x_0([N\mathbf{x}]) - b) \cap \mathbb{Z}^d}$, by the convexity of ξ_p

and Lemma 3, we obtain

$$\begin{aligned}
& \sum_{b \in \mathbb{Z}^d} \sum_{a_1, a_2 \in \mathbb{Z}^d} \sum_k' \exp \{ -c_{12} [\|k\| + \|k - a_1\| + \|k - a_2\|] + \\
& -\xi_p(k) - \xi_p(k - a_1) - \xi_p(k - a_2) - \xi_p([Nx_1] - b) + \xi_p([Nx_1] - x_0([N\mathbf{x}])) \\
& -\xi_p([Nx_2] - b - a_1) + \xi_p([Nx_2] - x_0([N\mathbf{x}])) - \xi_p([Nx_3] - b - a_2) + \xi_p([Nx_3] - x_0([N\mathbf{x}])) \} \\
& \leq \sum_{b \in \mathbb{Z}^d} \sum_{a_1, a_2 \in \mathbb{Z}^d} \sum_k' \exp \{ -c_{12} [\|k\| + \|k - a_1\| + \|k - a_2\|] + \\
& \quad -[\varphi_{p, [N\mathbf{x}]}(b + k) - \varphi_{p, [N\mathbf{x}]}(x_0([N\mathbf{x}]))] \} \\
& \leq \sum_{b \in \mathbb{Z}^d} \sum_{a_1, a_2 \in \mathbb{Z}^d} \sum_k' \exp \left\{ -c_{12} [\|k\| + \|k - a_1\| + \|k - a_2\|] - \frac{c_2}{N} \|x_0([N\mathbf{x}]) - (b + k)\|^2 \right\}.
\end{aligned} \tag{116}$$

Thus, for $\|x_0([N\mathbf{x}]) - b\| > N^\alpha$, denoting by $y = y(b, a_1, a_2) \in \mathbb{R}^d$ the minimizing point of the convex function

$$w(z) := \frac{c_2}{N} \|x_0([N\mathbf{x}]) - (b + z)\|^2 + c_{12} [\|z\| + \|z - a_1\| + \|z - a_2\|], \tag{117}$$

if $\|y\| \geq \frac{N^\alpha}{2}$, then (116) is smaller than $e^{-c_{16} \frac{N^\alpha}{2}}$. On the other hand, if $\|y\| < \frac{N^\alpha}{2}$, then (116) is smaller than $e^{-c_{17} \frac{N^{2\alpha-1}}{4}}$. Therefore, setting $\alpha' := \alpha \wedge (2\alpha - 1)$, for sufficiently large value of N , we get

$$\begin{aligned}
& e^{\varphi_{p, [N\mathbf{x}]}(x_0([N\mathbf{x}]))} \sum_{b \in \mathbb{Z}^d : \|b - x_0([N\mathbf{x}])\| > N^\alpha} \tilde{h}_{t_1}^{\eta, K}([Nx_1] - b) \sum_{a_1, a_2 \in \mathbb{Z}^d} g_p^{\eta, K}(0, a_1, a_2) \times \\
& \times \prod_{i=2,3} \tilde{h}_{t_i}^{\eta, K}([Nx_i] - b - a_{i-1}) \leq e^{-c_{18} N^{\alpha'}}.
\end{aligned} \tag{118}$$

Moreover, for any $\beta \in (0, \frac{1}{2})$,

$$\begin{aligned}
& e^{\varphi_{p,[N\mathbf{x}]}(x_0([N\mathbf{x}]))} \sum_{b \in \mathbb{Z}^d : \|b - x_0([N\mathbf{x}])\| \leq N^\alpha} \tilde{h}_{t_1}^{\eta,K}([Nx_1] - b) \times \\
& \times \sum_{a_1 \in \mathbb{Z}^d : \|a_1\| > N^\beta} \sum_{a_2 \in \mathbb{Z}^d} g_p^{\eta,K}(0, a_1, a_2) \prod_{i=1,2} \tilde{h}_{t_{i+1}}^{\eta,K}([Nx_{i+1}] - b - a_i) \\
& \leq \sum_{b \in \mathbb{Z}^d : \|b - x_0([N\mathbf{x}])\| \leq N^\alpha} \sum_{a_1 \in \mathbb{Z}^d : \|a_1\| > N^\beta} \sum_{a_2 \in \mathbb{Z}^d} \times \\
& \times \sum_k^I e^{-\frac{c_2}{N} \|x_0([N\mathbf{x}]) - (b+k)\|^2 - c_{12}[\|k\| + \|k - a_1\| + \|k - a_2\|]} \\
& \leq c_{19} N^{2\alpha d} \sum_{a_1 \in \mathbb{Z}^d : \|a_1\| > N^\beta} e^{-c_{12}\|a_1\|} \leq e^{-c_{20}N^\beta}.
\end{aligned} \tag{119}$$

A similar inequality holds with a_1 and a_2 exchanged.

Notice that (94), (119) and (118) imply

$$\mathbb{P}_p[F(k_1; [N\mathbf{x}]) \cap F(k_2; [N\mathbf{x}]) \cap \{\|k_1 - k_2\| > N^\beta\}] \leq e^{-\varphi_{p,[N\mathbf{x}]}(x_0([N\mathbf{x}])) - c_{21}N^{\beta \wedge \alpha'}} \tag{120}$$

and so what stated in Remark 2.

Then we are left with the estimate of

$$\begin{aligned}
& \sum_{b \in \mathbb{Z}^d : \|b - x_0([N\mathbf{x}])\| \leq N^\alpha} e^{\xi_p([Nx_1] - x_0([N\mathbf{x}]))} \tilde{h}_{t_1}^{\eta,K}([Nx_1] - b) \times \\
& \times \sum_{\substack{a_1 \in \mathbb{Z}^d : \|a_1\| \leq N^\beta \\ a_2 \in \mathbb{Z}^d : \|a_2\| \leq N^\beta}} g_p^{\eta,K}(0, a_1, a_2) \prod_{i=1,2} e^{\xi_p([Nx_{i+1}] - x_0([N\mathbf{x}]))} \tilde{h}_{t_{i+1}}^{\eta,K}([Nx_{i+1}] - b - a_i).
\end{aligned} \tag{121}$$

Now we choose $\alpha = \frac{1}{2} + \varepsilon$ and $\beta < \frac{1}{2} - \varepsilon$ with $\varepsilon \in (0, \frac{1}{6})$. For $\|b - x_0([N\mathbf{x}])\| \leq N^\alpha$ and

$\|a_1\|, \|a_2\| \leq N^\beta$ we get

$$\begin{aligned}
& \xi_p([Nx_1] - b) - \xi_p([Nx_1] - x_0([N\mathbf{x}])) = -(\nabla \xi_p([Nx_1] - x_0([N\mathbf{x}])), b - x_0([N\mathbf{x}])) + \\
& \quad + \frac{1}{2}(b - x_0([N\mathbf{x}]), H_\xi([Nx_1] - x_0([N\mathbf{x}]); p)(b - x_0([N\mathbf{x}])) + O(N^{-\frac{1}{2}+3\varepsilon}) \\
& \xi([Nx_{i+1}] - b - a_i) - \xi_p([Nx_{i+1}] - x_0([N\mathbf{x}])) = -(\nabla \xi_p([Nx_{i+1}] - x_0([N\mathbf{x}])), b + a_i - x_0([N\mathbf{x}])) + \\
& \quad + \frac{1}{2}(b + a_i - x_0([N\mathbf{x}]), H_\xi([Nx_{i+1}] - x_0([N\mathbf{x}]); p)(b + a_i - x_0([N\mathbf{x}])) + O(N^{-\frac{1}{2}+3\varepsilon}) \\
& \quad = -(\nabla \xi_p([Nx_{i+1}] - x_0([N\mathbf{x}])), b + a_i - x_0([N\mathbf{x}])) + \\
& \quad + \frac{1}{2}(b - x_0([N\mathbf{x}]), H_\xi([Nx_{i+1}] - x_0([N\mathbf{x}]); p)(b - x_0([N\mathbf{x}])) + \\
& \quad + O(N^{-\frac{1}{2}+\varepsilon+\beta\vee(2\varepsilon)}) \quad i = 1, 2.
\end{aligned} \tag{122}$$

Moreover, since $\sum_{i=1,2,3} \nabla \xi_p([Nx_i] - x_0([N\mathbf{x}])) = 0$, for any $x \in \mathbb{R}^d$,

$$\sum_{i=1,2,3} (\nabla \xi_p([Nx_i] - x_0([N\mathbf{x}])), x) = 0. \tag{123}$$

Then,

$$\begin{aligned}
& \xi_p([Nx_1] - b) + \sum_{i=1,2} \xi([Nx_{i+1}] - b - a_i) - \varphi_{p,[N\mathbf{x}]}(x_0([N\mathbf{x}])) \\
& = - \sum_{i=1,2} (\nabla \xi_p([Nx_{i+1}] - x_0([N\mathbf{x}])), a_i) + \\
& \quad + \frac{1}{2}(b - x_0([N\mathbf{x}]), H_\varphi(x_0([N\mathbf{x}]), [N\mathbf{x}]; p)(b - x_0([N\mathbf{x}])) + O(N^{-\frac{1}{2}+\beta\vee(2\varepsilon)+\varepsilon}).
\end{aligned} \tag{124}$$

Hence, making use of (45), (121) becomes

$$\begin{aligned}
& \prod_{i=1,2,3} \frac{\tilde{\Lambda}_p\left(\frac{x_i - x_0(\mathbf{x})}{\|x_i - x_0(\mathbf{x})\|}, t_i\right)}{\sqrt{2\pi N^{d-1} \|x_i - x_0(\mathbf{x})\|^{d-1}}} \sum_{b \in \mathbb{Z}^d : \|b - x_0([N\mathbf{x}])\| \leq N^{\frac{1}{2}+\varepsilon}} e^{-\frac{1}{2}(b - x_0([N\mathbf{x}]), H_\varphi(x_0([N\mathbf{x}]), [N\mathbf{x}]; p)(b - x_0([N\mathbf{x}]))} \times \\
& \quad \times \sum_{\substack{a_1 \in \mathbb{Z}^d : \|a_1\| \leq N^\beta \\ a_2 \in \mathbb{Z}^d : \|a_2\| \leq N^\beta}} g_p^{\eta, K}(0, a_1, a_2) e^{\sum_{i=1,2} (\nabla \xi_p([Nx_{i+1}] - x_0([N\mathbf{x}])), a_i)} (1 + o(1)).
\end{aligned} \tag{125}$$

Furthermore, by (93) and (123),

$$\begin{aligned}
& \sum_{\substack{a_1 \in \mathbb{Z}^d : \|a_1\| \leq N^\beta \\ a_2 \in \mathbb{Z}^d : \|a_2\| \leq N^\beta}} g_p^{\eta, K} (0, a_1, a_2) e^{\sum_{i=1,2} (\nabla \xi_p([Nx_{i+1}] - x_0([N\mathbf{x}]), a_i))} \\
& \leq \sum_{\substack{a_1 \in \mathbb{Z}^d : \|a_1\| \leq N^\beta \\ a_2 \in \mathbb{Z}^d : \|a_2\| \leq N^\beta}} \sum_k^I e^{-\xi_p(k) - \sum_{i=1,2} [\xi_p(k - a_i) - (\nabla \xi_p([Nx_{i+1}] - x_0([N\mathbf{x}]), a_i)) - c_{12} [\|k\| + \sum_{i=1,2} \|k - a_i\|]]} \\
& \leq \sum_{\substack{a_1 \in \mathbb{Z}^d : \|a_1\| \leq N^\beta \\ a_2 \in \mathbb{Z}^d : \|a_2\| \leq N^\beta}} \sum_k^I e^{-\xi_p(k) - (\nabla \xi_p([Nx_1] - x_0([N\mathbf{x}]), k) - \sum_{i=1,2} [\xi_p(k - a_i) - (\nabla \xi_p([Nx_{i+1}] - x_0([N\mathbf{x}]), a_i - k))]} \times \\
& \times e^{-c_{12} [\|k\| + \sum_{i=1,2} \|k - a_i\|]}.
\end{aligned} \tag{126}$$

But,

$$\begin{aligned}
\xi_p([Nx_i] - (b + k)) &= \xi_p([Nx_i] - x_0([N\mathbf{x}])) - (\nabla \xi_p([Nx_i] - x_0([N\mathbf{x}]), b + k - x_0([N\mathbf{x}])) + \\
& \quad + \frac{1}{2} (b + k - x_0([N\mathbf{x}]), H_\xi([Nx_i] - x_0([N\mathbf{x}]); p) (b + k - x_0([N\mathbf{x}])) + O(N^{-\frac{1}{2} + 3\varepsilon})
\end{aligned} \tag{127}$$

and by (122) it follows that

$$\begin{aligned}
\xi_p(k) &\geq \xi_p([Nx_i] - (b + k)) - \xi_p([Nx_i] - b) \\
&= -(\nabla \xi_p([Nx_i] - x_0([N\mathbf{x}]), k) + O(N^{-\frac{1}{2} + \beta + \varepsilon})).
\end{aligned} \tag{128}$$

Then, since for any $x, y \in \mathbb{R}^d$, $(\nabla \xi_p(x), y) \leq \xi_p(y)$, from (126) it follows that there exists a positive constant c_{22} such that

$$\begin{aligned}
& \sum_{\substack{a_1 \in \mathbb{Z}^d : \|a_1\| \leq N^\beta \\ a_2 \in \mathbb{Z}^d : \|a_2\| \leq N^\beta}} g_p^{\eta, K} (0, a_1, a_2) e^{\sum_{i=1,2} (\nabla \xi_p([Nx_{i+1}] - x_0([N\mathbf{x}]), a_i))} \\
& \leq \sum_{k \in \mathbb{Z}^d} \sum_{\substack{a_1 \in \mathbb{Z}^d : \|a_1\| \leq N^\beta \\ a_2 \in \mathbb{Z}^d : \|a_2\| \leq N^\beta}} e^{-c_{22} [\|k\| + \sum_{i=1,2} \|k - a_i\|]}.
\end{aligned} \tag{129}$$

Hence, the r.h.s. of (126) is bounded by a finite constant and

$$\begin{aligned} & \sum_{\substack{a_1 \in \mathbb{Z}^d : \|a_1\| \leq N^\beta \\ a_2 \in \mathbb{Z}^d : \|a_2\| \leq N^\beta}} g_p^{\eta, K}(0, a_1, a_2) e^{\sum_{i=1,2} (\nabla \xi_p([Nx_{i+1}] - x_0([N\mathbf{x}])) , a_i)} \\ &= \sum_{a_1, a_2 \in \mathbb{Z}^d} g_p^{\eta, K}(0, a_1, a_2) e^{\sum_{i=1,2} (t_{i+1}, a_i)} (1 + o(1)). \end{aligned} \quad (130)$$

Finally, by (91),

$$\mathbb{P}_p[T^*([N\mathbf{x}])] \leq \sum_{i=1,2,3} \sum_{k \in \mathbb{Z}^d} e^{-\varphi_{p,[N\mathbf{x}]}(k) - c_{10} \|k - [Nx_i]\|}. \quad (131)$$

By (108), we need only to estimate

$$\mathbb{P}_p[T_{\frac{1}{2}+\varepsilon, N}^*(\mathbf{x})] \leq \sum_{i=1,2,3} \sum_{k \in \mathbb{Z}^d : \|k - x_0([N\mathbf{x}])\| \leq N^{\frac{1}{2}+\varepsilon}} e^{-\varphi_{p,[N\mathbf{x}]}(k) - c_{10} \|k - [Nx_i]\|}, \quad (132)$$

but, by Lemma 3,

$$\varphi_{p,[N\mathbf{x}]}(k) \geq \varphi_{p,[N\mathbf{x}]}(x_0([N\mathbf{x}])) + \frac{c_2}{N} \|k - x_0([N\mathbf{x}])\|^2. \quad (133)$$

Thus, there exists a positive constant c_{23} such that,

$$\mathbb{P}_p[T^*([N\mathbf{x}])] \leq N^{d(\frac{1}{2}+\varepsilon)} e^{-\varphi_{p,[N\mathbf{x}]}(x_0([N\mathbf{x}])) - c_{23}N}. \quad (134)$$

Collecting all the previous estimates, from (125), (28) and (130) we obtain

$$\mathbb{P}_p[E([N\mathbf{x}])] = \frac{(2\pi)^{\frac{d}{2}} N^{\frac{d}{2}}}{\sqrt{\det H_\varphi(x_0(\mathbf{x}), \mathbf{x}; p)}} \frac{\theta_p(\mathbf{x})}{(2\pi N^{d-1})^{\frac{3}{2}}} e^{-\varphi_{p,[N\mathbf{x}]}(x_0([N\mathbf{x}]))} (1 + o(1)), \quad (135)$$

with

$$\theta_p(\mathbf{x}) := \prod_{i=1,2,3} \frac{\tilde{\Lambda}_p\left(\frac{x_i - x_0(\mathbf{x})}{\|x_i - x_0(\mathbf{x})\|}, t_i(\mathbf{x})\right)}{\sqrt{\|x_i - x_0(\mathbf{x})\|^{d-1}}} \sum_{a_1, a_2 \in \mathbb{Z}^d} g_p^{\eta, K}(0, a_1, a_2) e^{\sum_{i=1,2} (t_{i+1}(\mathbf{x}), a_i)} \quad (136)$$

analytic function on X'_3 .

Therefore, by (84),

$$\begin{aligned}
& \mathbb{P}_p \left[F \left(\left[Nx_0(\mathbf{x}) + y\sqrt{N} \right]; [N\mathbf{x}] \right) \mid E([N\mathbf{x}]) \right] \\
&= \frac{\mathbb{P}_p \left[F \left(\left[Nx_0(\mathbf{x}) + y\sqrt{N} \right]; [N\mathbf{x}] \right) \right]}{\mathbb{P}_p[E([N\mathbf{x}])]} \\
&= \frac{\Theta_p(\mathbf{x}) \sqrt{\det H_\varphi(x_0(\mathbf{x}), \mathbf{x}; p)}}{(2\pi)^{\frac{d}{2}} \theta_p(\mathbf{x}) N^{\frac{d}{2}}} e^{-\frac{1}{2}(y, H_\varphi(x_0(\mathbf{x}), \mathbf{x}; p)y)} (1 + o(1)),
\end{aligned} \tag{137}$$

which gives the asymptotic estimate (14) with

$$\Phi_p(\mathbf{x}) := \frac{\Theta_p(\mathbf{x})}{\theta_p(\mathbf{x})} = \frac{\sum_{a_1, a_2, a_3 \in \mathbb{Z}^d} g_p^{\eta, K}(0; a_1, a_2, a_3) e^{\sum_{i=1,2,3} (t_i(\mathbf{x}), a_i)}}{\sum_{a_1, a_2 \in \mathbb{Z}^d} g_p^{\eta, K}(0, a_1, a_2) e^{\sum_{i=1,2} (t_{i+1}(\mathbf{x}), a_i)}}. \tag{138}$$

■

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